

# Smeared D0 charge and the Gubser-Mitra conjecture

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## Abstract

We relate a Dp or NS-brane with D0-brane charge smeared over its world-volume to the system with no D0-charge. This allows us to generalise Reall's partial proof of the Gubser-Mitra conjecture. We show explicitly for specific examples that the dynamical instability coincides with thermodynamic instability in the ensemble where the D0-brane charge can vary. We also comment on consistency checks of the conjecture for more complicated systems, using the example of the D4 with F1 and D0 charges smeared on its worldvolume.

## 1 Introduction

Black  $p$ -brane solutions have played a central role in many investigations in string theory. However, although large classes of exact solutions have been known for some time, the dynamical features of the general solutions have not yet been thoroughly explored. Early on, Gregory and Laflamme showed that uncharged [1] and certain charged [2]  $p$ -branes are unstable to perturbations which oscillate in the extended directions of the black brane with a wavelength longer than the characteristic curvature scale of the black brane solution. However, these early investigations were not pursued further until quite recently.

In [3, 4], Gubser and Mitra conjectured that the Gregory-Laflamme instability would occur if and only if the solution was locally thermodynamically unstable. This conjecture was motivated by the dual interpretation of the thermodynamics of the black brane in string theory in terms of a field theory on the brane worldvolume, and in particular by the fact that the black brane thermodynamics is extensive in the extended directions. A partial proof for this conjecture was given by Reall in [5], relating the existence of a Euclidean negative mode to the mode at the threshold of the dynamical instability. This argument applied for  $p$ -brane solutions carrying magnetic  $D - (p + 2)$ -form charges, encompassing most of the elementary NS and D-brane solutions in string theory. These arguments predicted the onset of dynamical

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instabilities at points where the specific heat changes sign; numerical evidence for the existence of these instabilities was obtained in [6, 7, 8].

In this paper, we wish to consider the extension of Reall’s argument [5] to study smeared branes; that is,  $p$ -brane solutions carrying an electric  $q$  form (or magnetic  $D - q$  form) charge where  $q < p + 2$ . We will show that there is a straightforward extension to the case where the smeared charge is a D0-brane charge (that is, an electric two-form charge). Strings carrying a D0-brane charge were first considered in [9], and we will build on insights from that work. The case of 2-branes carrying both D2 and D0-brane charges was considered in [10], where the dynamical stability boundary was computed numerically, and shown to agree with thermodynamic stability. We generalise Reall’s arguments and consider  $p$ -branes carrying both a magnetic  $D - (p + 2)$ -form charge and the electric two-form charge, demonstrating the link between thermodynamic and dynamic stability analytically. We also comment on more complicated brane systems with smeared D0 charge and exhibit a thermodynamic consistency condition that must be obeyed for the Gubser-Mitra relation to hold.

In IIA string theory, we can think of the D0-brane charge in M theory terms as momentum along the M theory circle. This implies that the M theory lifts of the solution describing a  $p$ -brane carrying both a magnetic  $D - (p + 2)$ -form charge and the D0-brane charge and the solution without D0-brane charge differ by a boost. Since the solutions are locally the same in 11 dimensions, we can translate the argument of [5] to a relation between a certain threshold mode of the solution with magnetic  $D - (p + 2)$ -form charge and D0-brane charge and a negative mode of the corresponding Euclidean solution. Thus, we can add D0-brane charge to any case studied in Reall’s argument ‘for free’. Assuming the Gregory-Laflamme instabilities are the only unstable modes of the dynamical system, this enables us to predict the stability boundary for these solutions: in particular, it implies it is independent of the boost parameter determining the D0-brane charge, as in [9]. For the D2-D0 case, the stability boundary we find in this way agrees with that found in [10].

One may lift more complicated IIA brane bound states to M theory and again add smeared D0 charge by boosting the M theory solution. Any classical instability in the original system will remain in the system with D0-charge. Hence, if the Gubser-Mitra conjecture is correct, the system without D0 charge has a thermodynamic instability independent of the D0-brane charge. We explicitly check this for the D4-F1-D0 system and confirm this holds true for thermodynamic ensembles where the D0 brane charge is allowed to vary.

This use of the M theory perspective develops from a series of previous insights. It was observed in [11] that an ansatz for smeared black holes existed in which the equations were independent of the black hole’s charge under a two-form field strength. This observation was used in [9] to argue that the dynamical stability of the smeared D0-brane was independent of charge.<sup>1</sup> It was realised in [13, 14] that this simplicity in the ansatz was attributable to the fact that the charge could be thought of as the M theory boost, and this observation was exploited in [13] to explicitly construct the unstable mode.

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<sup>1</sup>Dynamical stability for a different class of smeared charged solutions was studied in [12].

Note that as we find that both the threshold unstable mode and the Euclidean negative mode can be carried through to the D0-brane charged case, we are arguing that the Gubser-Mitra conjecture continues to hold for these cases, contrary to the claims in [9]. As discussed in these cases in [13, 10], the conjecture is made consistent provided one chooses a thermodynamic ensemble which allows the smeared charge to vary. This is related to the fact that the smeared charge redistributes over the world-volume in the dynamical instability.

The structure of the remainder of this paper is fairly simple. In the next section, we will review the relation between the threshold unstable mode associated with dynamical instability and the negative mode associated with thermodynamic instability found in [5] for single  $p$ -branes. In section 3, we will then describe how we can change to an M theory description of the solutions, apply a boost, and reduce to IIA again to obtain a corresponding relation for solutions which carry an additional D0 brane charge. In section 3.1, we will discuss specific cases of  $Dp$ -D0 systems and check the agreement with previous work. In section 4 we discuss more general brane bound states with smeared D0 charge and show that the Gubser-Mitra conjecture again implies the thermodynamic stability boundary must be independent of the D0 charge, verifying this for the example of D4-F1-D0. Finally in section 5, we conclude with a short discussion of open questions and future directions.

## 2 Connecting thermodynamic and dynamical instability

In this section, we review the argument of [5] relating thermodynamic and dynamical instability. Recall considered black  $p$ -brane solutions of the action

$$S = \int d^D x \sqrt{-\hat{g}} \left( -\hat{R} + \frac{1}{2}(\partial\phi)^2 + \frac{1}{2n!} e^{a\phi} F_{(n)}^2 \right), \quad (1)$$

where  $n = D - (p + 2)$ , and the  $p$ -branes were taken to be magnetically charged under  $F_{(n)}$ . The cases we will be interested in are the F1, D2, D4, and NS5-brane solutions in type IIA supergravity (so  $D = 10$ ). For each of these we can reduce the IIA supergravity action to the above form. For the F1 string,  $F_{(7)}$  is the dual of the Kalb-Ramond 3-form and  $a = -1$ , for the NS5 brane  $F_{(3)}$  is the Kalb-Ramond 3-form and  $a = 1$ , while for the D2, D4 branes  $F_{(6,4)}$  is a RR field strength and  $a = (p - 3)/2$ .

Later, to consider the extension to include D0-brane charge, we also include in the action a two-form field strength  $F_{(2)}$ , so the total action we consider is

$$S = \int d^D x \sqrt{-\hat{g}} \left( -\hat{R} + \frac{1}{2}(\partial\phi)^2 + \frac{1}{2n!} e^{a\phi} F_{(n)}^2 + \frac{1}{4} e^{-\phi/2} F_{(2)}^2 \right). \quad (2)$$

In the black  $p$ -brane solutions, the spacetime coordinates can be naturally decomposed into coordinates  $(t, x^i)$ ,  $i = 1, \dots, p$  in the directions along the brane, a radial coordinate  $r$ , and coordinates  $x^m$  on a  $8 - p$  sphere in the transverse space. The solutions are invariant under  $ISO(p)$  translations and rotations in the  $x^i$  and  $SO(9 - p)$

rotations on the sphere. We also use a coarser division of the  $D$ -dimensional coordinates  $x^a$ , into the worldvolume directions  $x^i$  and the other directions, collectively  $x^\mu$  (we reserve the notation  $x^M$  for 11d coordinates).

To relate dynamical and thermodynamic stability, we are interested in considering on the one hand the dynamical perturbations of this solution, and on the other hand, the eigenfunctions of the operator appearing in the quadratic action for fluctuations around the Euclidean solution in the Euclidean path-integral approach to the thermodynamics.

In [5], the perturbation was studied in a different conformal frame,

$$g_{ab} = \exp\left(-\frac{(7-p)}{4a}\phi\right) \hat{g}_{ab}. \quad (3)$$

The perturbed metric and dilaton are  $g_{ab} = \bar{g}_{ab} + h_{ab}$ ,  $\phi = \bar{\phi} + \delta\phi$ . For s-wave perturbations, the perturbation of the magnetic field strength can be set to zero,  $\delta F_{(n)} = 0$ . The perturbation of the metric and dilaton are assumed to be plane waves in the spatial worldvolume directions,  $h_{ab} = e^{i\mu_i x^i} H_{ab}$ ,  $\delta\phi = e^{i\mu_i x^i} f$ . Assuming that the perturbation is longitudinal, it can be taken to be non-zero only in the  $x^\mu$  directions by a choice of gauge,  $H_{i\mu} = H_{ij} = 0$ .

Our real interest is in dynamical instabilities—that is, in solutions of the linearised equations of motion which grow exponentially in time  $H_{\mu\nu} \sim e^{\Omega t}$ . However, to make the connection to thermodynamic instability, we will focus on the threshold unstable mode, which appears at the critical value of  $\mu_i$  such that  $\Omega = 0$ . We will assume that the existence of such a threshold unstable mode for non-zero  $\mu_i$  indicates the existence of a real instability at longer wavelengths. We thus assume that the remaining functions  $H_{\mu\nu}, f$  are functions only of  $r$ . That is, we consider small perturbations that move us in the space of static solutions, breaking only the symmetry in the spatial worldvolume directions.

There is then a second-order equation for the dilaton perturbation  $f$ ,

$$\begin{aligned} & -\nabla^2 f + 2\beta \partial_\mu \bar{\phi} \partial^\mu f + H^{\mu\nu} \nabla_\mu \nabla_\nu \bar{\phi} - \beta H^{\mu\nu} \partial_\mu \bar{\phi} \partial_\nu \bar{\phi} + \partial_\mu \bar{\phi} \nabla_\nu \left( H^{\mu\nu} - \frac{1}{2} H^\rho_\rho \bar{g}^{\mu\nu} \right) \\ & - \frac{a}{2(n-1)!} e^{(\alpha+\beta)\bar{\phi}} \left( H^{\mu\nu} \bar{F}_{\mu\rho_1 \dots \rho_{n-1}} \bar{F}_\nu{}^{\rho_1 \dots \rho_{n-1}} - \frac{\alpha+\beta}{n} \bar{F}^2 f \right) = \lambda f, \end{aligned} \quad (4)$$

where  $\lambda = -\mu^2$ ,  $\alpha = a - (p-2)(7-p)/2a$ , and  $\beta = 2(7-p)/a$ . The metric perturbations satisfy a second-order equation in the  $x^\mu$  space,

$$\begin{aligned} & -\nabla^2 H_{\mu\nu} + 2\nabla_{(\mu} \nabla^\rho H_{\nu)\rho} - \nabla_\mu \nabla_\nu H^\rho_\rho + 2R_{\rho(\mu} H^\rho_{\nu)} + 2R_{(\mu|\rho\sigma|\nu)} H^{\rho\sigma} \\ & - \beta \left( 2\nabla_{(\mu} H_{\nu)}^\rho - \nabla^\rho H_{\mu\nu} \right) \partial_\rho \bar{\phi} + 2\beta \nabla_\mu \nabla_\nu f - 4(k + \beta^2) \partial_{(\mu} \bar{\phi} \partial_{\nu)} f \\ & + \frac{1}{(n-1)!} e^{\alpha+\beta\bar{\phi}} \left( (n-1) H^{\rho\sigma} \bar{F}_{\mu\rho\lambda_1 \dots \lambda_{n-2}} \bar{F}_{\nu\sigma}{}^{\lambda_1 \dots \lambda_{n-2}} - \bar{F}_{\mu\lambda_1 \dots \lambda_{n-1}} \bar{F}_\nu{}^{\lambda_1 \dots \lambda_{n-1}} (\alpha + \beta) f \right) = \lambda H_{\mu\nu}, \end{aligned} \quad (5)$$

where  $k = 1/2 - 9(7 - p)^2/2a^2$ , and in addition two constraints coming from the  $\mu i$  and  $i j$  components of the metric perturbation equations,

$$Y_\mu = \nabla_\nu H_\mu^\nu - \beta H_\mu^\nu \partial_\nu \bar{\phi} - 2f(k + \beta^2) \partial_\nu \bar{\phi} = 0, \quad (6)$$

$$Z = H_\rho^\rho - 2\beta f = 0. \quad (7)$$

The above equations define a solution of the linearised perturbation equations about a given black  $p$ -brane solution. This perturbation is independent of  $t$  and depends on the spatial worldvolume coordinates as  $e^{i\mu_i x^i}$ . The core of the argument of [5] was to show that this could be related to an normalisable off-shell negative mode of the corresponding Euclidean solution, which is independent of both  $t$  and the  $x^i$ .

The Euclidean version of the black brane solution is a candidate saddle-point for the Euclidean path-integral for the canonical ensemble,

$$Z = \int d[g]d[\phi]d[F]e^{-I[g,\phi,F]}, \quad (8)$$

where  $I$  is the Euclidean action, and the path integral is taken over all smooth Riemannian geometries with matter fields  $\phi$ ,  $F$ , with all fields periodic with period  $\beta = T^{-1}$  in imaginary time. Expanding this action about the saddle point,

$$I[g] = I_0[\bar{g}, \bar{\phi}, \bar{F}] + I_2[\bar{g}, \bar{\phi}, \bar{F}; \delta g, \delta \phi, \delta F], \quad (9)$$

the quadratic part of the action will be of the form

$$I_2 = \int d^D x \delta\psi_\alpha \Delta_{\alpha\beta} \delta\psi_\beta, \quad (10)$$

where we use  $\delta\psi_\alpha$  to denote collectively the fluctuations  $(\delta g_{ab}, \delta\phi, \delta F_{(n)})$ , and  $\Delta_{\alpha\beta}$  is a second-order differential operator involving the background fields  $(\bar{g}, \bar{\phi}, \bar{F})$ . If there is a normalisable negative mode in the small perturbations around the solution, that is, if there is a solution of

$$\Delta_{\alpha\beta} \delta\psi_\beta = \lambda \delta\psi_\alpha \quad (11)$$

for  $\lambda < 0$ , then the solution should be interpreted as an instanton rather than a genuine saddle-point approximation for the canonical ensemble [15, 5]. In [5], it was shown, following [16], that for the black  $p$ -brane solutions, negative specific heat implied the existence of a negative mode, and it was argued that the converse should also be true.

In [5], it was shown that if we take the ansatz

$$\delta g_{ab} = H_{ab}, \quad \delta\phi = f, \quad \delta F = 0 \quad (12)$$

for the fluctuation around the saddle-point, then the eigenvalue equation (11) will reduce precisely to the equations (4,5) above. Furthermore, (6,7) are necessary conditions for the negative mode to be normalisable; they correspond to an appropriate

choice of gauge in which the eigenvalue equation can really be written in the form (4,5). Thus, the conditions for the existence of a normalisable negative mode are precisely the same set of equations (4-7) which determine a threshold unstable mode. The two problems were also shown to involve the same boundary conditions [5]. Since  $\delta F = 0$  for these static linear modes, and they reduce to negative modes of the Euclidean action, the instability is at fixed  $n$ -form charge. The appropriate ensemble thermodynamically is then the canonical one, namely fixing the  $p$ -brane charge.

### 3 Adding D0-brane charge

We now want to consider adding D0-brane charge to the black  $p$ -brane solutions. Here, we will exploit the fact that these IIA supergravity solutions can be simply related to M theory; if we have a IIA supergravity solution with metric  $g_{ab}$ , dilaton  $\phi$  and  $n$ -form field strength  $F_{(n)}$ ,  $n = 3, 4, 6, 7$ , this can be used to construct a solution of M theory with

$$ds_{(11)}^2 = e^{4/3\phi} dz^2 + e^{-2/3\phi} ds_{(10)}^2, \quad (13)$$

and a four-form field strength  $H_{(4)}$  given by

$$H_{(4)} = \begin{cases} F_{(3)} \wedge dz, \\ F_{(4)}, \\ \star(F_{(6)} \wedge dz), \\ \star F_{(7)} \end{cases} \quad (14)$$

respectively. The 11d solution corresponding to the original IIA supergravity solution has a compact  $z$  direction, but let us consider the non-compact solution. We can then boost it in the  $t - z$  plane, defining coordinates

$$\begin{aligned} t' &= t \cosh \beta + z \sinh \beta, \\ z' &= z \cosh \beta + t \sinh \beta. \end{aligned} \quad (15)$$

If we then compactify the  $z'$  direction, we can define a new IIA background by

$$ds_{(11)}^2 = e^{4/3\phi'} (dz' + A)^2 + e^{-2/3\phi'} ds_{(10)}'^2, \quad (16)$$

which will carry a D0-brane charge under the two-form field strength  $F_{(2)} = dA$ . We will refer to this combined operation as a twist.

Starting with a black  $p$ -brane solution, this operation will give a D0-brane charge which is smeared over all the spatial worldvolume directions, as the  $ISO(p)$  symmetry in these directions is preserved. The threshold unstable mode of the original black  $p$ -brane geometry discussed in the previous section corresponds to going a small distance along a new branch of solutions which are not translationally invariant in the spatial worldvolume directions. We can apply the above boost + recompactification to this branch of solutions; hence, there is a corresponding threshold unstable mode for the resulting black  $p$ -brane with a smeared D0-brane charge. For the special case of an uncharged black string, this charge independence of the threshold mode was

observed in [9] using the observations of [11]. This twist argument provides a general understanding of that result (as previously observed in [14, 13]).

If we take the ansatz given in the previous section and use this twist to translate it into a threshold unstable mode for the black  $p$ -brane with non-zero D0-brane charge, we will find that the non-zero components of the metric perturbation are still of the form  $\delta g_{ab} = e^{i\mu_i x^i} H'_{ab}$ , and  $\delta\phi = e^{i\mu_i x^i} f'$ , but starting from the ansatz given above, we will have  $H'_{ij}$  non-zero. We will also have a perturbation  $\delta A = e^{i\mu_i x^i} a$ , but we still have  $\delta F_{(n)} = 0$ . The expressions for  $H'_{ab}$ ,  $f'$  and  $a$  in terms of  $H_{\mu\nu}$  and  $f$  depend on the background  $\bar{g}_{\mu\nu}$  and  $\bar{\phi}$ , but are straightforward to derive; we will not give them explicitly. Note that the critical wavelength will still have the same value  $\mu$ , independent of the D0-brane charge parameter  $\beta$ . Thus, the existence of this threshold unstable mode is independent of the D0-brane charge for any of the black  $p$ -brane solutions with smeared D0-brane charge.<sup>2</sup>

The fact that the critical wavelength is independent of  $\beta$  is somewhat surprising. We expect that the BPS smeared D0-brane solution will be marginally stable, but this result indicates that there will be an instability at wavelengths longer than  $\mu$  as soon as we move away from extremality. In [13], these instabilities were studied explicitly, and it was found that they grow in time as  $e^{\tilde{\Omega}t}$ , where  $\tilde{\Omega} \propto 1/\cosh\beta$ , so  $\tilde{\Omega} \rightarrow 0$  as  $\beta \rightarrow \infty$ , consistent with the expected stability of the BPS solution.

We should now consider if there is a corresponding Euclidean negative mode. The Euclidean negative mode of the black  $p$ -brane solution will define a corresponding  $z$ -independent negative mode of the M theory solution. The IIA supergravity action in (9,10) can be thought of as a consistent truncation of the full 11d supergravity action to a  $z$ -independent sector, so any eigenvector of (11) will also be an eigenvector of the corresponding operator in the full theory.

One might think that the Euclidean analogue of (15) would be a rotation in the  $\tau - z$  plane; that is, in addition to the analytic continuation  $t \rightarrow i\tau$ , we might imagine also making the analytic continuation  $\beta \rightarrow i\gamma$ . However, recall that our desire here is to use the Euclidean path integral as a tool to elucidate the thermodynamic properties of a Lorentzian solution carrying electric (D0-brane) charge. It was argued in [17] that to describe the thermodynamics of electrically charged solutions, one should not make an analytic continuation of the electric field; one should accept that the electric field will be imaginary in the Euclidean section. In [17] this was important to maintaining invariance under electric-magnetic duality; the usual Hodge duality in Euclidean space will relate real magnetic to imaginary electric fields. In the same way, here, we will interpret the Euclidean version of the black  $p$ -brane with D0-brane charge obtained by analytically continuing  $t \rightarrow i\tau$ , but keeping  $\beta$  real as the appropriate saddle-point for the path integral in the canonical ensemble.<sup>3</sup> From the 11d point of view, this

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<sup>2</sup>The  $p$ -branes which we can treat by this argument are the F1, D2, D4 and NS5-branes in IIA. The D6 is not included in this discussion, as it corresponds to a KK monopole in M theory, so we cannot take the  $z$  coordinate non-compact in the 11d solution. Recall also that the existence and critical wavelength of the instability will depend on the  $p$ -brane charge.

<sup>3</sup>One can avoid discussing imaginary fields by regarding the D0-brane charge as a magnetic charge under the dual 8-form field strength. The threshold unstable mode can also be dualised to rewrite it as a threshold unstable mode of the magnetic solution, which will be related to a real Euclidean

saddle point has a complex metric; this is similar to the discussion of rotating black hole solutions in [18].

Since the redefinition (15) is just a coordinate transformation, albeit now a complex one, it does not change the fact that the solution of (11) is a negative mode. Since the original solution was independent of both  $t$  and  $z$ , this will be  $z'$ -independent, and we can reduce it to obtain a negative mode around the saddle-point constructed from the Euclidean  $Dp$ -smeared D0 solution in the IIA theory. The explicit form of this negative mode is again given by taking the threshold unstable mode, dropping the dependence on the spatial worldvolume coordinates, and analytically continuing  $t \rightarrow i\tau$ . Thus, the existence of the negative mode is also independent of the D0-brane charge, and these two instabilities remain correlated.

As before,  $\delta F_{(n)} = 0$ , but we now have  $\delta A \neq 0$ . Following [5], it seems natural to interpret this Euclidean negative mode as indicating an instability in the ensemble where we fix the  $n$ -form charge, but allow the D0-brane charge to vary, as in [13, 10]. This is also in accord with the original argument of [3, 4], who concluded that the dynamical instability should be linked to thermodynamic instability in the ensemble where all extensive parameters are allowed to vary. We will not attempt to generalise the argument of [16, 5] to show that thermodynamic instability in this sense implies the existence of a negative mode in this more general case. Instead, we will now turn to the consideration of specific examples, and verify that in all cases the region of thermodynamic instability in this ensemble coincides with the region where this negative mode exists.

### 3.1 Examples

We would like to now illustrate this general discussion with some specific examples, and explicitly demonstrate the connection between the dynamical instability and thermodynamic instability. The simplest case to consider is when we start with an uncharged black  $p$ -brane solution, so the twist gives us just a non-extremal black D0-brane smeared over  $p$  transverse directions. This case was considered in [9, 13], where it was observed that the dynamical instability is independent of the charge. The metric in this case is (we give solutions in Einstein frame)

$$ds^2 = -H^{-7/8} f dt^2 + H^{1/8} (f^{-1} dr^2 + r^2 d\Omega_{8-p}^2 + d\vec{x}^2), \quad (17)$$

and the one-form gauge potential and dilaton are

$$A = \coth \beta (1 - H^{-1}) dt, \quad e^{2\phi} = H^{3/2}, \quad (18)$$

where

$$H = 1 + \frac{r_0^{7-p} \sinh^2 \beta}{r^{7-p}}, \quad f = 1 - \frac{r_0^{7-p}}{r^{7-p}}. \quad (19)$$

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negative mode on the Euclidean saddle-point solution. However, this dualising conceals the 11d origin of the simple behaviour of the instability.



We can read off the mass, charge and thermodynamic parameters from this solution,

$$M = \frac{\Omega_{8-p} V_p}{16\pi G} r_0^{7-p} ((8-p) + (7-p) \sinh^2 \beta), \quad (20)$$

$$Q_0 = \frac{\Omega_{8-p} V_p}{16\pi G} (7-p) r_0^{7-p} \sinh \beta \cosh \beta, \quad (21)$$

$$S = \frac{\Omega_{8-p} V_p}{4G} r_0^{8-p} \cosh \beta, \quad T = \frac{(7-p)}{4\pi r_0 \cosh \beta}, \quad \Phi_0 = \tanh \beta. \quad (22)$$

In [9], the charge was regarded as fixed, and the thermodynamic stability was studied by considering the specific heat,  $C_Q = (\frac{\partial M}{\partial T})$ . This changes sign near (but a finite distance from) extremality for  $p < 5$ , which was interpreted as an indication of thermodynamic stability, violating the Gubser-Mitra conjecture.

However, as we have discussed in the previous section, we should consider the thermodynamic ensemble which allows the D0 charge to fluctuate. That is, we should also consider the isothermal permittivity,  $\epsilon_T = (\frac{\partial Q}{\partial \Phi})_T$ , which probes the thermodynamic stability under changes of the charge. This is negative, indicating an instability, in precisely the region where the specific heat becomes positive, so there is indeed a thermodynamic instability for all values of the charges. Explicitly [6],

$$C_Q = -\frac{\Omega_{8-p} V_p}{4G} r_0^{8-p} \cosh \beta \frac{(7-p) + (9-p) \cosh 2\beta}{(7-p) - (5-p) \cosh 2\beta} \quad (23)$$

$$\epsilon_T = \frac{(7-p) \Omega_{8-p} V_p}{32\pi G} r_0^{7-p} \cosh^2 \beta ((7-p) - (5-p) \cosh 2\beta), \quad (24)$$

so if  $p < 5$ ,  $C_Q > 0$  for sufficiently large  $\beta$ , but we never have  $C_Q > 0$  and  $\epsilon_T > 0$ , indicating that the smeared D0-brane solution is always unstable in the grand canonical ensemble, as expected from the existence of the Euclidean negative mode obtained by our general argument.

It is interesting to note that these smeared D0-brane solutions are also related to the charged Dp-branes discussed in [5] by T-duality in the spatial worldvolume directions, but this relation is not helpful for understanding the stability. The dynamical instability depends on the worldvolume directions, so it would not be expected to be invariant. Rather these unstable gravity modes should presumably be thought of as unstable stringy winding modes in the T-dual theory. The thermodynamic stability fails to be invariant for a subtler reason: the thermodynamic quantities are invariant under T-duality, but the appropriate ensemble is the canonical one (fixed charge) for the Dp-brane solution, but the grand canonical one (charge allowed to vary) for the smeared D0-brane solution [13].

The smeared D0-brane case is thus seen to be consistent with the Gubser-Mitra conjecture, once we understand the proper application of the conjecture in this case. It is more interesting to consider a case with a non-trivial boundary of stability, which will provide a sharp test of the conjecture. The case of D2-branes with smeared D0-branes was studied in [10], where a numerical analysis of the Lorentzian linear dynamics indicated consistency of the Gubser-Mitra conjecture. We may now show this analytically. We start from a solution with D2-brane charge,

$$ds^2 = Z^{-5/8} [-f dt^2 + dx_1^2 + dx_2^2] + Z^{3/8} (f^{-1} dr^2 + r^2 d\Omega_6^2), \quad (25)$$

with gauge field and dilaton

$$F_{(6)} = 5r_0^5 \sinh \gamma \cosh \gamma \epsilon_{S^6}, \quad e^{2\phi} = H^{1/2}, \quad (26)$$

where

$$Z = 1 + \frac{r_0^5 \sinh^2 \gamma}{r^5}, \quad f = 1 - \frac{r_0^5}{r^5}, \quad (27)$$

and  $\epsilon_{S^6}$  is the volume form on the unit six-sphere, corresponding to the metric  $d\Omega_6^2$ . Following [5] and our general discussion above, we are writing the D2-brane charge as a magnetic charge under  $F_{(6)}$ . We can obtain a solution with D2-brane and smeared D0-brane charge by applying the twist. This gives a solution

$$ds^2 = H^{-5/8} [-f dt^2 + H Z^{-1} (dx_1^2 + dx_2^2)] + H^{3/8} (f^{-1} dr^2 + r^2 d\Omega_6^2), \quad (28)$$

with a one-form gauge field and dilaton

$$A = \frac{\cosh^2 \gamma \sinh \beta \cosh \beta}{\sinh^2 \gamma \cosh^2 \beta + \sinh^2 \beta} (1 - H^{-1}) dt, \quad e^{2\phi} = H^{3/2} Z^{-1}, \quad (29)$$

and the D2-brane gauge field is unchanged,

$$F_{(6)} = 5r_0^5 \sinh \gamma \cosh \gamma \epsilon_{S^6}, \quad (30)$$

where

$$H = Z \cosh^2 \beta - f \sinh^2 \beta = 1 + \frac{r_0^5 (\sinh^2 \gamma \cosh^2 \beta + \sinh^2 \beta)}{r^5}, \quad (31)$$

and  $f$ ,  $Z$  are as in (27). This solution was written in a different form by Gubser in [10], where the D2-brane charge was written in terms of a gauge field  $A_3$ ; the twist then produces a non-zero NS two-form  $B_2$ . He also uses a function  $D = H Z^{-1}$ , and parameters  $\alpha$ ,  $\theta$  related to the ones used here by

$$\sinh^2 \alpha = \sinh^2 \gamma \cosh^2 \beta + \sinh^2 \beta \quad (32)$$

and

$$\sin^2 \theta = \frac{\cosh^2 \gamma \sinh^2 \beta}{\sinh^2 \gamma \cosh^2 \beta + \sinh^2 \beta}. \quad (33)$$

This implies

$$\sinh^2 \gamma = \sinh^2 \alpha \cos^2 \theta. \quad (34)$$

Our general twist argument predicts that the instability will be independent of  $\beta$ : this is in agreement with the results of Gubser [10], who finds a thermodynamic stability boundary at

$$\sinh \gamma = \sinh \alpha \cos \theta = \frac{1}{\sqrt{3}}, \quad (35)$$

which is confirmed by numerical studies of the dynamical instability. In this case, the thermodynamic instability is at fixed D2-brane charge, but allowing the D0-brane charge to vary, corresponding to the fact that  $\delta F_{(6)} = 0$  but  $\delta A \neq 0$  in the ansatz for the threshold unstable mode and the negative mode.

Another interesting case to consider is a D4-brane with smeared D0-brane charge. The solution is

$$ds^2 = H_0^{-7/8} H_4^{-3/8} (-f dt^2 + H_0 d\vec{x}^2) + H_0^{1/8} H_4^{5/8} (f^{-1} dr^2 + r^2 d\Omega_4^2), \quad (36)$$

with gauge fields and dilaton

$$A = \coth \beta (1 - H_0^{-1}) dt, \quad F_{(4)} = 3r_0^3 \sinh \alpha \cosh \alpha \epsilon_{S^4}, \quad e^{2\phi} = H_0^{3/2} H_4^{-1/2}, \quad (37)$$

where

$$H_0 = 1 + \frac{r_0^3 \sinh^2 \beta}{r^3}, \quad H_4 = 1 + \frac{r_0^3 \sinh^2 \alpha}{r^3}, \quad f = 1 - \frac{r_0^3}{r^3}, \quad (38)$$

and  $\epsilon_{S^4}$  is the volume form on the unit four-sphere. The parameter dependence in this case is simpler: from the 11d point of view, the D4-brane comes from a reduction along a worldvolume direction in the M5-brane, whereas to obtain a D2-brane we had to smear the M2-brane over a transverse direction. From the 10d point of view, the simplicity is related to the fact that the BPS D4-D0 is a marginally bound system. The thermodynamic parameters are

$$M = \frac{\Omega_4 V_4}{16\pi G} r_0^3 [4 + 3 \sinh^2 \alpha + 3 \sinh^2 \beta], \quad (39)$$

$$Q_4 = \frac{\Omega_4}{16\pi G} 3r_0^3 \sinh \alpha \cosh \alpha, \quad (40)$$

$$Q_0 = \frac{\Omega_4 V_4}{16\pi G} 3r_0^3 \sinh \beta \cosh \beta, \quad (41)$$

$$S = \frac{\Omega_4 V_4}{4G} r_0^4 \cosh \alpha \cosh \beta, \quad T = \frac{3}{4\pi r_0 \cosh \alpha \cosh \beta}, \quad (42)$$

$$\Phi_0 = \tanh \beta, \quad \Phi_4 = \tanh \alpha. \quad (43)$$

We again want to calculate the Hessian of the entropy with respect to  $M$  and  $Q_0$ , holding  $Q_4$  fixed. Evaluating this Hessian, we find

$$\det H = \frac{(\frac{4}{9}\pi)^2 (16\pi G)^2 (\sinh^4 \alpha - 1)}{\Omega_4^2 V_4^2 r_0^4 (4 \sinh^2 \alpha \sinh^2 \beta + 5 \sinh^2 \alpha + 5 \sinh^2 \beta + 4)}. \quad (44)$$

The boundary of thermodynamic stability is at  $\det H = 0$ , so we see that it is independent of  $\beta$ , as expected from our general argument. The system is unstable for  $\sinh^2 \alpha < 1$  and stable for  $\sinh^2 \alpha > 1$ . In figure 1, we plot this stability boundary in the  $Q_4, Q_0$  plane for fixed  $M$ . We plot the square roots of the charges to give the BPS bound a similar form to the D2-D0 case studied in [10].

We can also consider F1 and smeared D0 or NS5 and smeared D0. We will not describe these two cases in detail, as they add no really new elements. The F1-D0 case is qualitatively similar to the D4-D0 case discussed above, while the NS5-D0 is not as interesting, as there is no stability boundary: the system is always both thermodynamically and dynamically unstable.

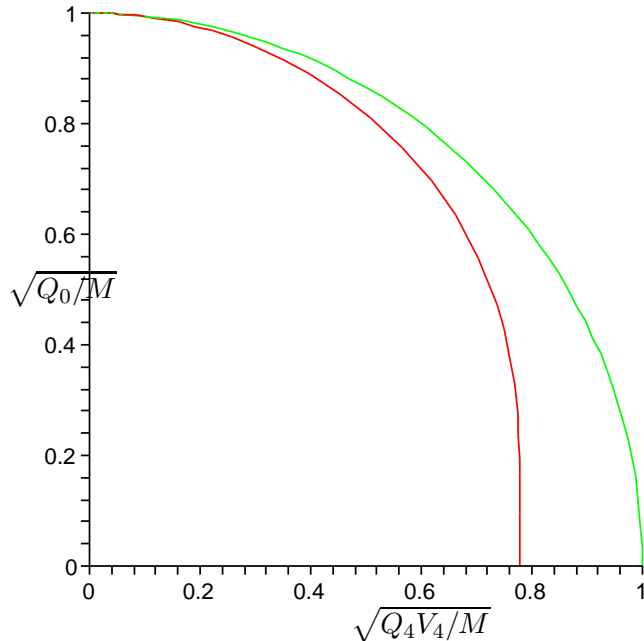


Figure 1: The boundary of stability for the D4-smeared D0 system, as a function of  $\sqrt{Q_4 V_4/M}$  and  $\sqrt{Q_0/M}$ . The system is stable to the right of the line. The circle indicates the BPS bound.

## 4 Adding D0 charge to general brane bound states

Provided a translationally invariant brane bound state can be lifted to M theory, much of the discussion above still applies. Take a IIA system  $X$ , lift to M theory and boost. Reducing then gives a smeared  $X$ -D0 system. For example D4-F1, with the F1 charge smeared on the D4, lifts to M5-M2 and after a twisted reduction gives the D4-F1-D0 system.

The principal difference from our previous discussion is that for general systems  $X$  we have no argument to relate classical and thermodynamical stability for  $X$ . However, what remains true is that any classical instability of  $X$  will lift to an instability in M theory. The twisted reduction boosts on the 11-circle and thus does not effect this instability mode. Hence the  $X$ -D0 system will exhibit the same instability for any D0 charge; as before, the critical wavelength for the instability will be independent of the D0 brane charge parameter  $\beta$ , but the instability will become weaker as we approach extremality.

This provides a consistency check of the Gubser-Mitra conjecture. If it holds for both  $X$  and  $X$ -D0, then the classical stability of  $X$  and  $X$ -D0 will be connected to the thermodynamic stability of these systems. This implies that the thermodynamic instability boundary for  $X$ -D0 must be independent of the D0 charge parameter, where we consider thermodynamic stability with the D0 charge allowed to vary.

Consider for example D4-F1-D0. The metric for the D4-F1 is found in [19] and using the twisted reduction method we may add D0-brane charge. Doing so gives

thermodynamic relations similar in form to those of D4-D0 considered above, except there is an additional parameter  $\theta$  related to the presence of the F1's,

$$M = \frac{\Omega_4 V_4}{16\pi G} r_0^3 [4 + 3 \sinh^2 \alpha + 3 \sinh^2 \beta], \quad (45)$$

$$Q_4 = \frac{\Omega_4}{16\pi G} 3r_0^3 \sinh \alpha \cosh \alpha \cos \theta, \quad (46)$$

$$Q_1 = \frac{\Omega_4 V_4}{16\pi G} 3r_0^3 \sinh \alpha \cosh \alpha \sin \theta, \quad (47)$$

$$Q_0 = \frac{\Omega_4 V_4}{16\pi G} 3r_0^3 \sinh \beta \cosh \beta, \quad (48)$$

$$S = \frac{\Omega_4 V_4}{4G} r_0^4 \cosh \alpha \cosh \beta, \quad T = \frac{3}{4\pi r_0 \cosh \alpha \cosh \beta}, \quad (49)$$

$$\Phi_0 = \tanh \beta, \quad \Phi_1 = \tanh \alpha \sin \theta, \quad \Phi_4 = \tanh \alpha \cos \theta. \quad (50)$$

In considering the Gubser-Mitra conjecture, the thermodynamic ensemble whose stability is related to dynamic stability will have the D4 charge fixed. For the F1 charge, the same logic suggests that the ensemble where the F1 charge is fixed will be related to dynamic stability against perturbations which vary only in the directions along the F1 string, while the ensemble where the F1 charge is allowed to vary will be related to perturbations which vary in the directions perpendicular to the F1 string in the D4 worldvolume. We will consider both ensembles, firstly with fixed F1 charge and secondly with it varying.

For the first ensemble we consider the Hessian of the entropy with respect to  $M$  and  $Q_0$ . This simply gives the same answer as for the D4-D0 system of the previous section. Hence the stability boundary is indeed independent of  $\beta$ , the D0 charge parameter.

For the second ensemble, where we allow the F1 and D0 charge to vary, the stability is given by the Hessian of the entropy with respect to  $M$ ,  $Q_0$  and  $Q_1$ , holding just  $Q_4$  fixed. One finds

$$\det H = \frac{(\frac{4}{9}\pi)^3 (16\pi G)^3 \cosh \alpha \cosh \beta (\sinh^2 \alpha \cos^2 \theta - 1)}{\Omega_4^3 V_4^3 r_0^6 (4 \sinh^2 \alpha \sinh^2 \beta + 5 \sinh^2 \alpha + 5 \sinh^2 \beta + 4)}. \quad (51)$$

and hence the stability region is given simply by  $\sinh^2 \alpha \cos^2 \theta > 1$ , and again passes our consistency check being independent of the D0 charge parameter  $\beta$ .

Note that we have parametrized the D4 and F1 charges in terms of  $\alpha, \theta$ , in the same way that Gubser [10] parametrized the D2-D0 system. This is convenient for studying the stability in the ensemble where we fix both the D4 and F1 charges. However, in the ensemble where we allow the F1 charge to vary, it would be more natural to use a description analogous to the one we used in the previous section for D2-D0; that is, to have independent D4 and F1 charge parameters  $\gamma, \delta$  such that

$$\sinh^2 \alpha = \sinh^2 \gamma \cosh^2 \delta + \sinh^2 \delta \quad (52)$$

and

$$\sin^2 \theta = \frac{\cosh^2 \gamma \sinh^2 \delta}{\sinh^2 \gamma \cosh^2 \delta + \sinh^2 \delta}. \quad (53)$$

Then we have

$$\sinh^2 \gamma = \sinh^2 \alpha \cos^2 \theta, \quad (54)$$

and the stability boundary is independent of the F1 charge parameter  $\delta$ , as in the discussion of the D2-D0 case.

This independence can be understood by relating this system to a system with smeared D0 branes by U-duality. From D2-D0, a T-duality in a transverse direction followed by S-duality followed by another T-duality in a transverse direction will give us the D4-F1 solution. Since these T-dualities require us to smear the system over additional transverse directions, the D4-F1 system is really U-dual to a system with D2-branes smeared over two transverse directions and D0-branes smeared over all four directions. The stability of this system thus need not be related to that of the D2-D0 system we studied in the previous section, and in fact, the stability boundary has changed: it's at  $\sinh^2 \gamma = 1$  for D4-F1, but it was at  $\sinh^2 \gamma = 1/3$  for D2-D0. Nonetheless, we can regard the D2-brane smeared over two transverse directions as another system  $X$  to which we are adding smeared D0-brane charge by twisting. The threshold unstable modes of the D4-F1 system are related to the threshold unstable modes of the smeared D2-D0 system which only depend on the directions along the D2-brane's worldvolume, so these threshold unstable modes should be independent of the charge parameter  $\delta$ . As in the argument we gave earlier in this section, this then predicts that the thermodynamic stability boundary will be independent of the F1 charge parameter  $\delta$ .

## 5 Conclusions

We have argued that the Gubser-Mitra conjecture holds for a number of  $p$ -brane systems with smeared D0-branes, by extending the argument given in [5]. These arguments explicitly show the classical stability agrees with the Euclidean negative mode. We also showed in a number of examples that this agrees with the boundary of local thermodynamic stability of the ensemble where the D0-charge is allowed to vary, again agreeing with Gubser & Mitra's idea that all extensive worldvolume quantities should be allowed to fluctuate. Furthermore, we have found that in those cases where there is a boundary separating unstable from stable solutions (namely D2-D0, D4-D0 and F1-D0), this boundary takes a simple form: it is independent of the parameter  $\beta$  associated with the D0-brane charge. These results were obtained by relating the solutions with D0-brane charge to the plain  $p$ -brane solutions discussed in [5] via an 11d M theory solution, where the parameter  $\beta$  is interpreted as parameterising a boost in the additional direction.

We have also argued that if the Gubser-Mitra conjecture is correct, more complicated brane systems with smeared D0 charge must have thermodynamic instability boundaries independent of this D0 charge, since we have shown the dynamical instabilities are independent of this. This provides a consistency check of the conjecture. We have verified that it is satisfied in the D4-F1-D0 system.

This discussion seems to be very special to the case of D0-branes, and it is not clear how to extend it to other cases. However, the resulting relationship between

the threshold unstable mode and the negative mode for the solutions with smeared D0-brane charge has much the same flavour as in the case without smeared charge. The negative mode is just obtained by dropping the dependence on the spatial world-volume coordinates in the threshold unstable mode and analytically continuing to the Euclidean solution,  $t \rightarrow i\tau$ . It is the complicated form for the ansatz for the threshold unstable mode allowing us to reduce the equations of motion to a form independent of  $\beta$  that is the key insight obtained from the twist.

We can also obtain some further results for other cases by T-duality. We have already noted that the T-duality relating the D0-brane smeared over  $p$  directions to the  $Dp$ -brane is not interesting for understanding the instability, as the instability depends on these directions. However, if we consider a D0-brane smeared over  $p + q$  directions for  $q > 0$ , we can consider an instability which is oriented so it only depends on  $q$  of the spatial coordinates. We can then T-dualise in the other  $p$  directions to obtain a  $Dp$ -brane smeared over  $q$  transverse directions. The instability will be invariant under this transformation, and the grand canonical ensemble will still be the relevant one for the smeared  $Dp$ -brane. We can therefore use our results above to conclude that any  $Dp$ -brane smeared over transverse directions will always be both dynamically and thermodynamically unstable, and that the wavelength of the dynamical instability will be independent of the charge parameter  $\beta$ .

Similarly, we can T-dualise the D2-D0 system on one direction to obtain an orthogonally intersecting pair of D1's smeared over the two worldvolume directions, and we can T-dualise the D4-D0 over one or two directions to obtain orthogonal D3-D1 and D2-D2 systems. These will all have the same stability properties as the cases studied here.

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## References

- [1] R. Gregory and R. Laflamme, “Black strings and p-branes are unstable,” *Phys. Rev. Lett.* **70** (1993) 2837–2840, [hep-th/9301052](#).
- [2] R. Gregory and R. Laflamme, “The Instability of charged black strings and p-branes,” *Nucl. Phys.* **B428** (1994) 399–434, [hep-th/9404071](#).
- [3] S. S. Gubser and I. Mitra, “Instability of charged black holes in anti-de Sitter space,” [hep-th/0009126](#).
- [4] S. S. Gubser and I. Mitra, “The evolution of unstable black holes in anti-de Sitter space,” *JHEP* **08** (2001) 018, [hep-th/0011127](#).
- [5] H. S. Reall, “Classical and thermodynamic stability of black branes,” *Phys. Rev. D* **64** (2001) 044005, [hep-th/0104071](#).

- [6] T. Hirayama, G.-W. Kang, and Y.-O. Lee, “Classical stability of charged black branes and the Gubser-Mitra conjecture,” *Phys. Rev. D* **67** (2003) 024007, [hep-th/0209181](#).
- [7] S. S. Gubser and A. Ozakin, “Universality classes for horizon instabilities,” *JHEP* **05** (2003) 010, [hep-th/0301002](#).
- [8] G. Kang and J. Lee, “Classical stability of black D3-branes,” *JHEP* **03** (2004) 039, [hep-th/0401225](#).
- [9] P. Bostock and S. F. Ross, “Smeared branes and the Gubser-Mitra conjecture,” *Phys. Rev. D* **70** (2004) 064014, [hep-th/0405026](#).
- [10] S. S. Gubser, “The Gregory-Laflamme instability for the D2-D0 bound state,” [hep-th/0411257](#).
- [11] T. Harmark and N. A. Obers, “Black holes on cylinders,” *JHEP* **05** (2002) 032, [hep-th/0204047](#).
- [12] O. Sarbach and L. Lehner, “Critical bubbles and implications for critical black strings,” *Phys. Rev. D* **71** (2005) 026002, [hep-th/0407265](#).
- [13] O. Aharony, J. Marsano, S. Minwalla, and T. Wiseman, “Black hole - black string phase transitions in thermal 1+1 dimensional supersymmetric Yang-Mills theory on a circle,” *Class. Quant. Grav.* **21** (2004) 5169–5192, [hep-th/0406210](#).
- [14] T. Harmark and N. A. Obers, “New phases of near-extremal branes on a circle,” *JHEP* **09** (2004) 022, [hep-th/0407094](#).
- [15] D. J. Gross, M. J. Perry, and L. G. Yaffe, “Instability of flat space at finite temperature,” *Phys. Rev. D* **25** (1982) 330–355.
- [16] B. F. Whiting and J. W. York, “Action principle and partition function for the gravitational field in black hole topologies,” *Phys. Rev. Lett.* **61** (1988) 1336.
- [17] S. W. Hawking and S. F. Ross, “Duality between electric and magnetic black holes,” *Phys. Rev. D* **52** (1995) 5865–5876, [hep-th/9504019](#).
- [18] J. D. Brown, E. A. Martinez, and J. W. York, “Complex Kerr-Newman geometry and black hole thermodynamics,” *Phys. Rev. Lett.* **66** (1991) 2281–2284.
- [19] T. Harmark, “Supergravity and space-time non-commutative open string theory,” *JHEP* **07** (2000) 043, [hep-th/0006023](#).